

# On the Perturbation-Iteration Algorithm for System of Fractional Differential Equations

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## Abstract

In this study, perturbation-iteration algorithm, namely PIA, is applied to solve some types of system of fractional differential equations (FDEs) for the first time. To illustrate the efficiency of the method, numerical solutions are compared with the results published in the literature by considering some FDEs. The results confirm that the PIA is powerful, simple and reliable method for solving system of nonlinear fractional differential equations.

**Keywords:** Fractional-integro differential equations, Caputo fractional derivative, Initial value problems, Perturbation-Iteration Algorithm. AMS 2010: 34A08, 34A12.

## 1 Introduction

As an important mathematical branch investigating the properties of derivatives and integrals, the history of the fractional calculus is nearly as old as classical integer order analysis. Even though there is a long period of time since the beginning of fractional calculus, it could not find practical applications for many years. However, in last decades, it has found a place in various areas such as control theory, Senol et al. (2014), viscoelasticity, Yu and Lin. (1998), electrochemistry, Oldham (2010) and electromagnetic, Heaviside (2008).

The evolution of the symbolic computation programs such as Matlab and Mathematica is one of the driving forces behind this increased usage. Various multidisciplinary problems could be expressed by the help of fractional derivatives and integrals. For example, in Hamamci, (2007). The most important descriptions of fundamentals of fractional calculus have been studied by Mainardi, (1997) and Podlubny, (1998). Existence and uniqueness of the solutions has also been studied by Yakar and Koksal (2012) and the references therein.

Parallel to the studies in applied sciences, system of fractional differential equations (FDEs) allowed scientists to describe and model various important and useful physical problems.

The number of differential equations whose solution can not be found analytically. That situations appear in FDEs more than the other types of differential equations. In this case, as the study of algorithms using numerical approximation for the problems of mathematical analysis, the field of numerical analysis is used for approximate solutions of FDEs. In recent years, a significant effort has been expended to propose numerical methods for this purpose. These methods include, fractional variational iteration method, Wu and Lee, (2010); Guo and Mei (2011), homotopy perturbation method, Abdulaziz et al. (2008); He (2012); Momani and Odibat, (2007); Zhang et al. (2014) and fractional differential transform method (Momani et al., 2007; Arikoglu and Ozkol, 2009; El-Sayed et al., 2014).

In this study, we have applied the previously developed method PIA to obtain approximate solutions for some system of FDEs. Our method is suitable for a broad class of equations and does not require special assumptions and restrictions.

In the literature, there exists a few fractional derivative definitions of an arbitrary order. Two mostly used of them are the Riemann-Liouville and Caputo fractional derivatives. The two definitions are quite similar but have different order of evaluation of derivation.

Riemann-Liouville fractional integration of order  $\alpha$  is defined by:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \quad x > 0. \quad (1)$$

The following two equations are defined as Riemann-Liouville and Caputo fractional derivatives of order  $\alpha$ , respectively.

$$D^\alpha f(x) = \frac{d^m}{dx^m} (J^{m-\alpha} f(x)) \quad (2)$$

$$D_*^\alpha f(x) = J^{m-\alpha} \left( \frac{d^m}{dx^m} f(x) \right). \quad (3)$$

where  $m-1 < \alpha < m$  and  $m \in \mathbf{N}$ .

Due to the appropriateness of the initial conditions, fractional definition of Caputo is often used in recent years.

**Definition 1.1** *The fractional derivative of  $u(x)$  in the Caputo sense is defined as*

$$D_*^\alpha u(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} u^{(m)}(t) dt & , m-1 < \alpha < m \\ \frac{d^m}{dx^m} u(x) & , \alpha = m \end{cases} \quad (4)$$

for  $m-1 < \alpha < m$ ,  $m \in \mathbf{N}$ ,  $x > 0$ ,  $u \in C_{-1}^m$ .

Also, we need here two of its basic properties.

**Lemma 1.2** *If  $m-1 < \alpha < m$ ,  $m \in \mathbf{N}$  and  $u \in C_\mu^m$ ,  $\mu-1$  then*

$$D_*^\alpha J^\alpha u(x) = u(x) \quad (5)$$

and

$$J^\alpha D_*^\alpha u(x) = u(x) - \sum_{k=0}^{m-1} u^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \quad (6)$$

After this introductory section, Section 2 is reserved to a brief review of the Perturbation-Iteration Algorithm PIA(1,1), in Section 3 some examples are presented to show the efficiency and simplicity of the algorithm. Finally the paper ends with a conclusion in Section 4.

## 2 Overview of the Perturbation-Iteration Algorithm PIA(1,1)

As one of the most practical subjects of physics and mathematics, differential equations create models for a number of problems in science and engineering to give an explanation for a better understanding of the events. Perturbation methods have been used for this purpose for over a century (Nayfeh, 2008; Jordan and Smith, 1987; Skorokhod et al., 2002). These methods could be used to search approximate solutions of integral equations, difference equations, integro-differential equations and partial differential equations.

But the main difficulty in the application of perturbation methods is the requirement of a small parameter or to install a small artificial parameter in the equation. For this reason, the obtained solutions are restricted by

a validity range of physical parameters. Therefore, to overcome the disadvantages come with the perturbation techniques, some methods have been suggested by several authors (He, 2001; Mickens, 1987, 2005, 2006; Cooper, 2002; Hu and Xiong, 2003; He, 2012; Wang and He, 2008; Iqbal and Javed, 2011; Iqbal et al., 2010).

Parallel to these studies, recently a new perturbation-iteration algorithm has been proposed by Aksoy, Pakdemirli and their co-workers, Aksoy and Pakdemirli (2010); Pakdemirli et al. (2011); Aksoy et al. (2012). A previous attempt of constructing root finding algorithms systematically, Pakdemirli and Boyacı(2007), Pakdemirli et al. (2007); Pakdemirli et al. (2008). finally guided to generalization of the method to differential equations also Aksoy and Pakdemirli, (2010); Pakdemirli et al. (2011); Aksoy et al. (2012). In the new technique, an iterative algorithm is established on the perturbation expansion. The method has been applied to first order equations Pakdemirli et al. (2011) and Bratu type second order equations, Aksoy and Pakdemirli, (2010) to obtain approximate solutions. Then the algorithms were tested on some nonlinear heat equations also Aksoy et al. (2012). Finally, the solutions of the Volterra and Fredholm type integral equations (Dolapci et al., 2013) and ordinary differential equation and systems, Şenol et al. (2013) have given by the present method.

In this study, the previously developed new technique is applied to some types of nonlinear fractional differential equations for the first time. To obtain the approximate solutions of equations, the most basic perturbation-iteration algorithm PIA(1,1) is employed by taking one correction term in the perturbation expansion and correction terms of only first derivatives in the Taylor series expansion, i.e.  $n = 1, m = 1$ .

Consider the following initial value problem.

$$F_k(D^\alpha u_k, u_j, \varepsilon, t) = 0 \quad (7)$$

$$k = 1, 2, \dots, K$$

$$j = 1, 2, \dots, K$$

$$u^{(k)}(0) = c_k, \quad k = 0, 1, 2, \dots, m-1, \quad m-1 < \alpha \leq m \quad (8)$$

where  $K$  is the number of the equation in the system and  $D^\alpha$  is the Caputo fractional derivative of order  $\alpha$ , which is defined by:

$$D^\alpha u(t) = J^{m-\alpha} \left( \frac{d^m}{dt^m} u(t) \right), \quad m-1 < \alpha < m, \quad m \in \mathbf{N} \quad (9)$$

As more clearly the system could be expressed by:

$$\begin{aligned} F_1 &= F_1(u_1^{(\alpha)}, u_1, u_2, \dots, u_k, \varepsilon, t) = 0 \\ F_2 &= F_2(u_2^{(\alpha)}, u_1, u_2, \dots, u_k, \varepsilon, t) = 0 \\ &\vdots \\ F_K &= F_K(u_k^{(\alpha)}, u_1, u_2, \dots, u_k, \varepsilon, t) = 0 \end{aligned} \quad (10)$$

In this method as  $\varepsilon$  is the artificially introduced perturbation parameter and we use only one correction term in the perturbation expansion.

$$\begin{aligned} u_{k,n+1} &= u_{k,n} + \varepsilon(u_c)_{k,n} \\ u'_{k,n+1} &= u'_{k,n} + \varepsilon(u'_c)_{k,n} \end{aligned} \quad (11)$$

where subscript  $n$  represents the  $n$ .iteration

Replacing (11) into (7) and writing in the Taylor Series expansion for only first order derivatives in the neighborhood of  $\varepsilon = 0$  gives

$$F_K = \sum_{m=0}^M \frac{1}{m!} \left[ \left( \frac{d}{d\varepsilon} \right)^m F_K \right]_{\varepsilon=0} \varepsilon^m, \quad k = 1, 2, \dots, K \quad (12)$$

for

$$\frac{d}{d\varepsilon} = \frac{\partial u_{k,n+1}^{(\alpha)}}{\partial \varepsilon} \frac{\partial}{\partial u_{k,n+1}^{(\alpha)}} + \sum_{j=1}^K \left( \frac{\partial u_{j,n+1}}{\partial \varepsilon} \frac{\partial}{\partial u_{j,n+1}} \right) + \frac{\partial}{\partial \varepsilon} \quad (13)$$

This equation is defined for the  $(n+1)$ .iteration equation

$$F_k \left( u_{k,n+1}^{(\alpha)}, u_{j,n+1}, \varepsilon, t \right) = 0 \quad (14)$$

Replacing (13) in (12) yields our iteration equation:

$$F_K = \sum_{m=0}^M \frac{1}{m!} \left[ \left( \frac{((u_c)_{k,n})^{(\alpha)}}{\partial \varepsilon} \frac{\partial}{\partial u_{k,n+1}^{(\alpha)}} + \sum_{j=1}^K (u_c)_{j,n} \frac{\partial}{\partial u_{j,n+1}} + \frac{\partial}{\partial \varepsilon} \right)^m F_K \right]_{\varepsilon=0}^m \varepsilon^m = 0, \quad (15)$$

$$k = 1, 2, \dots, K$$

All derivatives are calculated at  $\varepsilon = 0$ .

Beginning with an initial function  $u_0$ , first  $(u_c)_{k,n}$  's has been determined by the help of (15). Then using Eq. (11),  $(n+1)$ . iteration solution could be found Iteration process is repeated using (15) and (11) until achieving an acceptable result.

### 3 Application

**Example 3.1** Consider the following system of linear fractional differential equations Abdulaziz et al. 2008:

$$\begin{aligned} D^\alpha u_1(t) &= u_1(t) + u_2(t) \\ D^\beta u_1(t) &= -u_1(t) + u_2(t) \end{aligned} \quad (16)$$

$$0 < \alpha, \beta \leq 1$$

given with the initial conditions  $u_1(0) = 0$  and  $u_2(0) = 1$ . The known exact solutions for  $\alpha = \beta = 1$  are

$$u_1(t) = e^t \sin t \quad (17)$$

and

$$u_2(t) = e^t \cos t \quad (18)$$

Eq. (16) can be rearranged in the following form with adding and subtracting  $u'_{1,n}(t)$  and  $u'_{2,n}(t)$  to the equation:

$$\varepsilon \frac{d^\alpha u_1(t)}{dt^\alpha} + u'_{1,n}(t) - \varepsilon u'_{1,n}(t) - \varepsilon u_{1,n}(t) - \varepsilon u_{2,n}(t) = 0 \quad (19)$$

$$\varepsilon \frac{d^\beta u_2(t)}{dt^\beta} + u'_{2,n}(t) - \varepsilon u'_{2,n}(t) + \varepsilon u_{1,n}(t) - \varepsilon u_{2,n}(t) = 0 \quad (20)$$

where  $\varepsilon$  is a small parameter. According to the iteration formula for

$$F(u'_1, u_1, \varepsilon) = \frac{1}{\Gamma(1-\alpha)} \varepsilon \int_0^t \frac{u'_1(s)}{(t-s)^\alpha} ds + u'_{1,n}(t) - \varepsilon u'_{1,n}(t) - \varepsilon u_{1,n}(t) - \varepsilon u_{2,n}(t) \quad (21)$$

$$F(u'_2, u_2, \varepsilon) = \frac{1}{\Gamma(1-\beta)} \varepsilon \int_0^t \frac{u'_2(s)}{(t-s)^\beta} ds + u'_{2,n}(t) - \varepsilon u'_{2,n}(t) + \varepsilon u_{1,n}(t) - \varepsilon u_{2,n}(t) \quad (22)$$

terms in equation (15) become

$$\begin{aligned} F &= u'_{1,n}(t), \quad F_{u_1} = 0, \quad F_{u'_1} = 1, \\ F_\varepsilon &= -u'_{1,n}(t) - u_{1,n}(t) - u_{2,n}(t) + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'_{1,n}(s)}{(t-s)^\alpha} ds \end{aligned} \quad (23)$$

for the iteration formula

$$u'_1(t) + \frac{F_{u_1}}{F_{u'_1}} u_1(t) = -\frac{F_\varepsilon + \frac{F}{\varepsilon}}{F_{u'_1}} \quad (24)$$

and terms in equation (15)

$$\begin{aligned} F &= u'_{2,n}(t), \quad F_{u_2} = 0, \quad F_{u'_2} = 1, \\ F_\varepsilon &= -u'_{2,n}(t) + u_{1,n}(t) - u_{2,n}(t) + \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{u'_{2,n}(s)}{(t-s)^\beta} ds \end{aligned} \quad (25)$$

for the iteration formula

$$u'_2(t) + \frac{F_{u_2}}{F_{u'_2}} u_2(t) = -\frac{F_\varepsilon + \frac{F}{\varepsilon}}{F_{u'_2}} \quad (26)$$

Notice that inserting the small parameter  $\varepsilon$  as a coefficient of the integral term simplifies the system and makes it easy to solve.

After writing these terms in the iteration formulas, the system gives the following differential equations.

$$u_{1,n}(t) + u_{2,n}(t) + \frac{(-1+\varepsilon)u'_{1,n}(t)}{\varepsilon} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'_{1,n}(s)}{(t-s)^\alpha} ds + (u_c)'_{1,n}(t) \quad (27)$$

and

$$-u_{1,n}(t) + u_{2,n}(t) + \frac{(-1+\varepsilon)u'_{2,n}(t)}{\varepsilon} = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{u'_{2,n}(s)}{(t-s)^\beta} ds + (u_c)'_{2,n}(t) \quad (28)$$

When employing the iteration formula (15), we start with an initial function compatible to the boundary condition and we determine coefficients from the boundary condition at each step. Beginning with the initial functions

$$u_{1,0}(t) = 0 \quad \text{and} \quad u_{2,0}(t) = 1$$

and using the iteration formula, we get the following successive approximate solutions at each step for  $n = 0, 1, 2, \dots$

$$\begin{aligned} u_{1,1}(t) &= t \\ u_{2,1}(t) &= 1+t \\ u_{1,2}(t) &= t \left( 2+t - \frac{t^{1-\alpha}}{\Gamma(3-\alpha)} \right) \\ u_{2,2}(t) &= 1+2t - \frac{t^{2-\beta}}{\Gamma(3-\beta)} \\ u_{1,3}(t) &= \frac{1}{3}t \left( 9+t(9+t) + \frac{3t^{2-2\alpha}}{\Gamma(4-2\alpha)} - \frac{9t^{1-\alpha}(3+t-\alpha)}{\Gamma(4-\alpha)} - \frac{3t^{2-\beta}}{\Gamma(4-\beta)} \right) \\ u_{2,3}(t) &= 1+3t - \frac{t^3}{3} + \frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + t^{2-2\beta} \left( \frac{t}{\Gamma(4-2\beta)} - \frac{t^\beta(9+t-3\beta)}{\Gamma(4-\beta)} \right) \end{aligned} \quad (29)$$

and so on. Following in this manner the fifth iteration solutions  $u_{1,5}(t)$  and  $u_{2,5}(t)$  were calculated. Due to brevity reasons, higher iterations are not given here. It is easy to calculate other iterations up to any order by the help of a symbolic calculation software such as Mathematica. In Figure 1. and 2. we compare our  $u_{1,5}(t)$  and  $u_{2,5}(t)$  solutions for different values of  $\alpha$  and  $\beta$  and in Figure 3. and 4. with the exact solution graphically. In Table 1. and 2. some of our iterations are compared with the exact solution. The results show that the proposed method can give successful approximations.

Table 1: Numerical results of Example 1. for some values of  $u_1(t)$ .

$\alpha = \beta = 1$							
$t$	$u_{1,1}(t)$	$u_{1,2}(t)$	$u_{1,3}(t)$	$u_{1,4}(t)$	$u_{1,5}(t)$	<i>Exact Solution</i>	<i>Absolute Error</i>
<b>0.0</b>	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
<b>0.1</b>	0.100000	0.110000	0.110333	0.110333	0.110332	0.110332	1.12698E-8
<b>0.2</b>	0.200000	0.240000	0.242666	0.242666	0.242656	0.242655	7.31405E-7
<b>0.3</b>	0.300000	0.390000	0.399000	0.399000	0.398919	0.398910	8.44622E-6
<b>0.4</b>	0.400000	0.559999	0.581333	0.581333	0.580991	0.580943	0.00004809
<b>0.5</b>	0.500000	0.750000	0.791666	0.791666	0.790625	0.790439	0.00018591
<b>0.6</b>	0.600000	0.960000	1.032000	1.031999	1.029408	1.028845	0.00056233
<b>0.7</b>	0.700000	0.190000	1.304333	1.304333	1.298731	1.297295	0.00143589
<b>0.8</b>	0.800000	1.440000	1.610666	1.610666	1.599744	1.596505	0.00323866
<b>0.9</b>	0.900000	1.710000	1.953000	1.952999	1.933316	1.926673	0.00664370
<b>1.0</b>	1.000000	2.000000	2.333333	2.333333	2.300000	2.287355	0.01264471

Table 2: Numerical results of Example 1. for some values of  $u_2(t)$ .

$\alpha = \beta = 1$							
$t$	$u_{2,1}(t)$	$u_{2,2}(t)$	$u_{2,3}(t)$	$u_{2,4}(t)$	$u_{2,5}(t)$	<i>Exact Solution</i>	<i>Absolute Error</i>
<b>0.0</b>	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	0.00000000
<b>0.1</b>	1.100000	1.099999	1.099666	1.099650	1.099649	1.099649	1.6274E-10
<b>0.2</b>	1.200000	1.200000	1.197333	1.197066	1.197055	1.197056	2.13559E-8
<b>0.3</b>	1.300000	1.300000	1.291000	1.289650	1.289568	1.289569	3.74045E-7
<b>0.4</b>	1.400000	1.400000	1.378666	1.374399	1.374058	1.374061	2.87222E-6
<b>0.5</b>	1.500000	1.500000	1.458333	1.447916	1.446874	1.446889	0.00001403
<b>0.6</b>	1.600000	1.600000	1.528000	1.506399	1.503807	1.503859	0.00005154
<b>0.7</b>	1.700000	1.700000	1.585666	1.545650	1.540047	1.540803	0.00015535
<b>0.8</b>	1.800000	1.800000	1.629333	1.561066	1.550144	1.550549	0.00040529
<b>0.9</b>	1.900000	1.900000	1.659999	1.547650	1.527966	1.528913	0.00094681
<b>1.0</b>	2.000000	2.000000	1.666666	1.500000	1.466666	1.468693	0.00202727

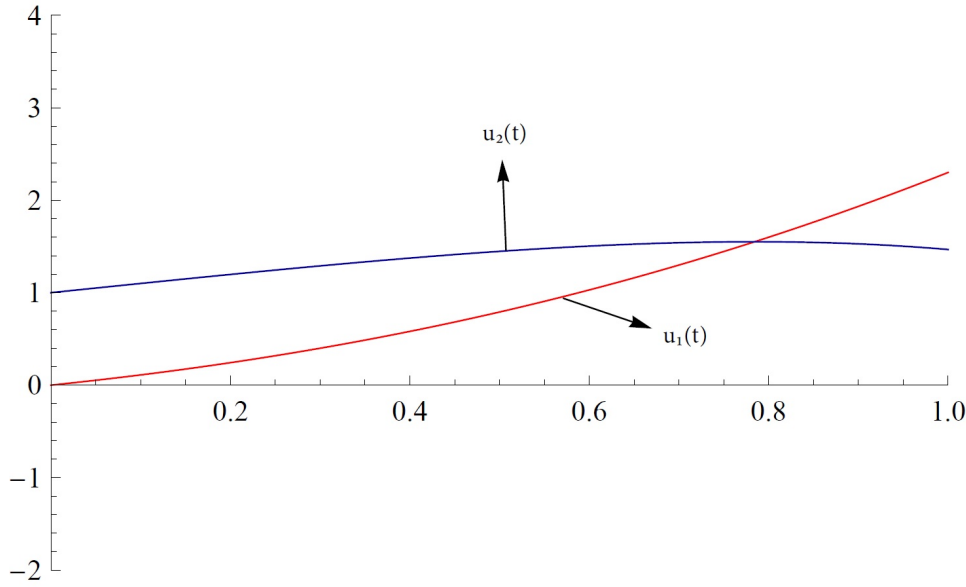


Figure 1: Comparison of the  $u_{1,5}(t)$  and  $u_{2,5}(t)$  of Example 1. for  $\alpha = \beta = 1$ .

**Example 3.2** For the second example consider the following system of nonlinear fractional differential equations Zurigat et al. 2010:

$$\begin{aligned} D^{\alpha_1} u_1(t) &= \frac{1}{2} u_1(t) \\ D^{\alpha_2} u_2(t) &= u_2(t) + u_1^2(t) \end{aligned} \quad (30)$$

$$0 < \alpha_1, \alpha_2 \leq 1$$

given with the initial conditions  $u_1(0) = 1$  and  $u_2(0) = 0$ . The exact solutions, when  $\alpha_1 = \alpha_2 = 1$ , are

$$u_1(t) = e^{\frac{t}{2}} \quad (31)$$

and

$$u_2(t) = te^t \quad (32)$$

In the system, if add and subtract  $u'_{1,n}(t)$  and  $u'_{2,n}(t)$  respectively, the system could be rewritten in the following form:

$$\varepsilon D^{\alpha_1} u_1(t) + u'_{1,n}(t) - \varepsilon u'_{1,n}(t) - \frac{1}{2} \varepsilon u_{1,n}(t) = 0 \quad (33)$$

$$\varepsilon D^{\alpha_2} u_2(t) + u'_{2,n}(t) - \varepsilon u'_{2,n}(t) - \varepsilon u_{2,n}(t) - \varepsilon u_{1,n}^2(t) = 0 \quad (34)$$

where  $\varepsilon$  is a small parameter. For

$$F(u'_1, u_1, \varepsilon) = \frac{1}{\Gamma(1-\alpha_1)} \varepsilon \int_0^t \frac{u'_{1,n}(s)}{(t-s)^{\alpha_1}} ds + u'_{1,n}(t) - \varepsilon u'_{1,n}(t) - \frac{1}{2} \varepsilon u_{1,n}(t) \quad (35)$$

$$F(u'_2, u_2, \varepsilon) = \frac{1}{\Gamma(1-\alpha_2)} \varepsilon \int_0^t \frac{u'_{2,n}(s)}{(t-s)^{\alpha_2}} ds + u'_{2,n}(t) - \varepsilon u'_{2,n}(t) - \varepsilon u_{2,n}(t) - \varepsilon u_{1,n}^2(t) \quad (36)$$

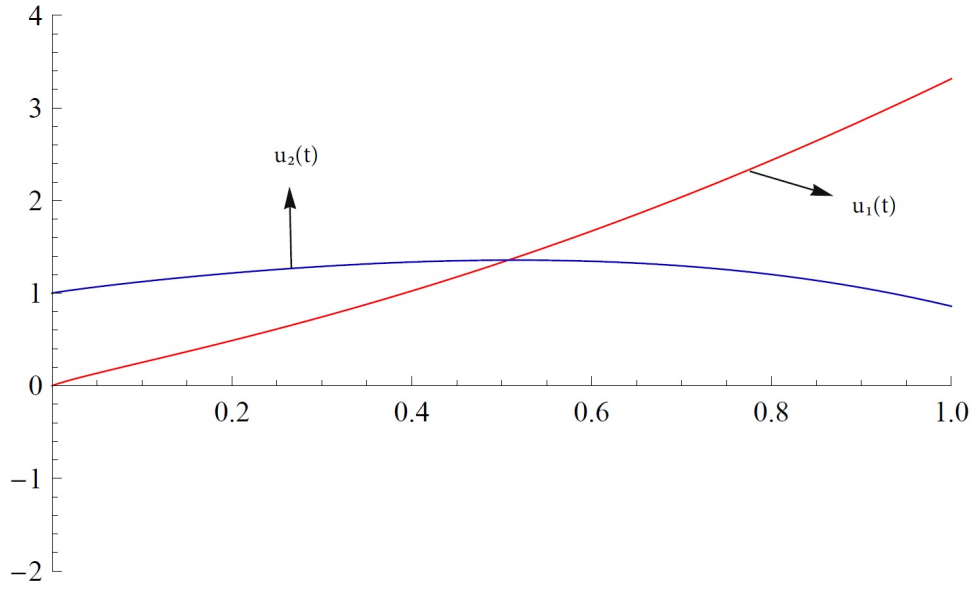


Figure 2: Comprasion of the  $u_{1,5}(t)$  and  $u_{2,5}(t)$  of Example 1. for  $\alpha = 0.7$  and  $\beta = 0.9$ .

and terms in equation (15) become

$$\begin{aligned} F &= u'_{1,n}(t), \quad F_{u_1} = 0, \quad F_{u'_1} = 1, \\ F_\varepsilon &= -u'_{1,n}(t) - \frac{u_{1,n}(t)}{2} + \frac{1}{\Gamma(1-\alpha_1)} \int_0^t \frac{u'_{1,n}(s)}{(t-s)^{\alpha_1}} ds \end{aligned} \quad (37)$$

and

$$\begin{aligned} F &= u'_{2,n}(t), \quad F_{u_2} = 0, \quad F_{u'_2} = 1, \\ F_\varepsilon &= -u'_{2,n}(t) - u_{2,n}(t) - u_{1,n}^2(t) + \frac{1}{\Gamma(1-\alpha_2)} \int_0^t \frac{u'_{2,n}(s)}{(t-s)^{\alpha_2}} ds \end{aligned} \quad (38)$$

After writing these terms in the iteration formula we obtain the following differential equations:

$$2 \left( \frac{\int_0^t (-s+t)^{-\alpha_1} u'_{1,n}(s) ds}{\Gamma(1-\alpha_1)} + (u_c)'_{1,n}(t) - \frac{(-1+\varepsilon)(u_{1,n})'(t)}{\varepsilon} \right) = u_{1,n}(t) \quad (39)$$

and

$$u_{2,n}(t) + u_{1,n}^2(t) + \frac{(-1+\varepsilon)(u_{2,n})'(t)}{\varepsilon} = \frac{\int_0^t (-s+t)^{-\alpha_2} (u_{2,n})'(s) ds}{\Gamma(1-\alpha_2)} + (u_c)'_{2,n}(t) \quad (40)$$

Beginning with the initial functions

$$u_{1,0}(t) = 1 \text{ and } u_{2,0}(t) = 0$$

and using the iteration formula, the following successive approximate solutions are obtained for  $n = 0, 1, 2, \dots$



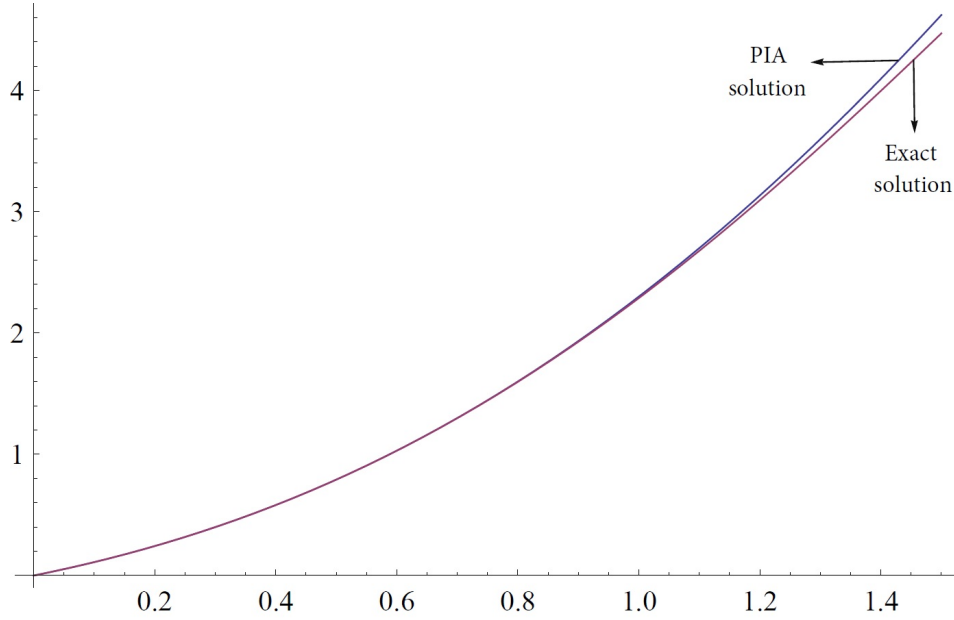


Figure 3: Comprasion of the exact solution and the PIA solution  $u_{1,5}(t)$  of Example 1. for  $\alpha = \beta = 1$ .

$$\begin{aligned}
u_{1,1}(t) &= 1 + \frac{t}{2} \\
u_{2,1}(t) &= t \\
u_{1,2}(t) &= \frac{1}{8} \left( 8 + 8t + t^2 + \frac{4t^{2-\alpha_1}}{\Gamma(2-\alpha_1)(-2+\alpha_1)} \right) \\
u_{2,2}(t) &= 2t + t^2 + \frac{t^3}{12} + \frac{t^{2-\alpha_2}}{\Gamma(2-\alpha_2)(-2+\alpha_2)} \\
u_{1,3}(t) &= \frac{1}{48} \left( 48 + 72t + 18t^2 + t^3 + \frac{24t^{3-2\alpha_1}}{\Gamma(4-2\alpha_1)} - \frac{24t^{2-\alpha_1}(9+t-3\alpha_1)}{\Gamma(4-\alpha_1)} \right) \\
u_{2,3}(t) &= 3t + 3t^2 + \frac{5t^3}{6} + \frac{t^4}{12} + \frac{t^5}{320} + \frac{t^{3-2\alpha_2}}{\Gamma(4-2\alpha_2)} - \frac{t^{5-2\alpha_1}}{4\Gamma(3-\alpha_1)^2(-5+2\alpha_1)} \\
&\quad - \frac{t^{3-\alpha_1}(4(40+3t(10+t)) + \alpha_1(-72-t(64+7t) + (8+t(8+t))\alpha_1))}{8\Gamma(6-\alpha_1)} \\
&\quad - \frac{t^{2-\alpha_2}(72+t(24+t) + 6\alpha_2(-7-t+\alpha_2))}{2\Gamma(5-\alpha_2)}
\end{aligned} \tag{41}$$

and so on. In the same manner the fourth iteration solutions  $u_{1,4}(t)$  and  $u_{2,4}(t)$  are calculated. Again we compared our results In Figures 5., 6., 7. and 8. and in Table 3. and 4. with the exact solutions.

## 4 Conclusion

In this paper we have applied previously developed numerical method so-called Perturbation-Iteration Algorithm (PIA) to find approximate solutions of some system of nonlinear Fractional Differential Equations for the first time. The numerical results obtained in this study show that method PIA is a remarkably successful numerical technique for solving system of FDEs. We expect that the present method could used to calculate the approximate solutions of other types of fractional differential equations such as fractional integro-differential equations and fractional partial differential equations. Our next study will be about these types of equations.

Table 3: Numerical results of Example 2. for some values of  $u_1(t)$ .

$\alpha_1 = \alpha_2 = 1$						
$t$	$u_{1,1}(t)$	$u_{1,2}(t)$	$u_{1,3}(t)$	$u_{1,4}(t)$	<i>Exact Solution</i>	<i>Absolute Error</i>
<b>0.0</b>	1.000000	1.000000	1.000000	1.000000	1.000000	0.0000000
<b>0.1</b>	1.050000	1.051250	1.051270	1.051271	1.051271	2.6260E-9
<b>0.2</b>	1.100000	1.105000	1.105166	1.105170	1.105170	8.4742E-8
<b>0.3</b>	1.150000	1.161250	1.161812	1.161833	1.161834	6.4897E-7
<b>0.4</b>	1.200000	1.220000	1.221333	1.221400	1.221402	2.7581E-6
<b>0.5</b>	1.250000	1.281250	1.283854	1.284016	1.284025	8.4896E-6
<b>0.6</b>	1.300000	1.345000	1.349500	1.349837	1.349858	0.0000213
<b>0.7</b>	1.350000	1.411250	1.418395	1.419021	1.419067	0.0000464
<b>0.8</b>	1.400000	1.480000	1.490666	1.491733	1.491824	0.0000913
<b>0.9</b>	1.450000	1.551250	1.566437	1.568146	1.568312	0.0001660
<b>1.0</b>	1.500000	1.625000	1.645833	1.648437	1.648721	0.0002837

Table 4: Numerical results of Example 2. for some values of  $u_2(t)$ .

$\alpha_1 = \alpha_2 = 1$						
$t$	$u_{2,1}(t)$	$u_{2,2}(t)$	$u_{2,3}(t)$	$u_{2,4}(t)$	<i>Exact Solution</i>	<i>Absolute Error</i>
<b>0.0</b>	0.000000	0.000000	0.000000	0.000000	0.000000	0.0000000
<b>0.1</b>	0.100000	0.110083	0.110505	0.110516	0.110517	2.4666E-7
<b>0.2</b>	0.200000	0.240666	0.244084	0.244272	0.244280	8.1286E-6
<b>0.3</b>	0.300000	0.392250	0.403929	0.404894	0.404957	0.0000635
<b>0.4</b>	0.400000	0.565333	0.593365	0.596453	0.596729	0.0002760
<b>0.5</b>	0.500000	0.760416	0.815852	0.823492	0.824360	0.0008683
<b>0.6</b>	0.600000	0.978000	1.074993	1.091043	1.0932712	0.0022277
<b>0.7</b>	0.700000	1.218583	1.374530	1.404661	1.409626	0.0049654
<b>0.8</b>	0.800000	1.482666	1.718357	1.770446	1.780432	0.0099863
<b>0.9</b>	0.900000	1.770750	2.110517	2.195074	2.213642	0.0185684
<b>1.0</b>	1.000000	2.083333	2.555208	2.685825	2.718281	0.0324559

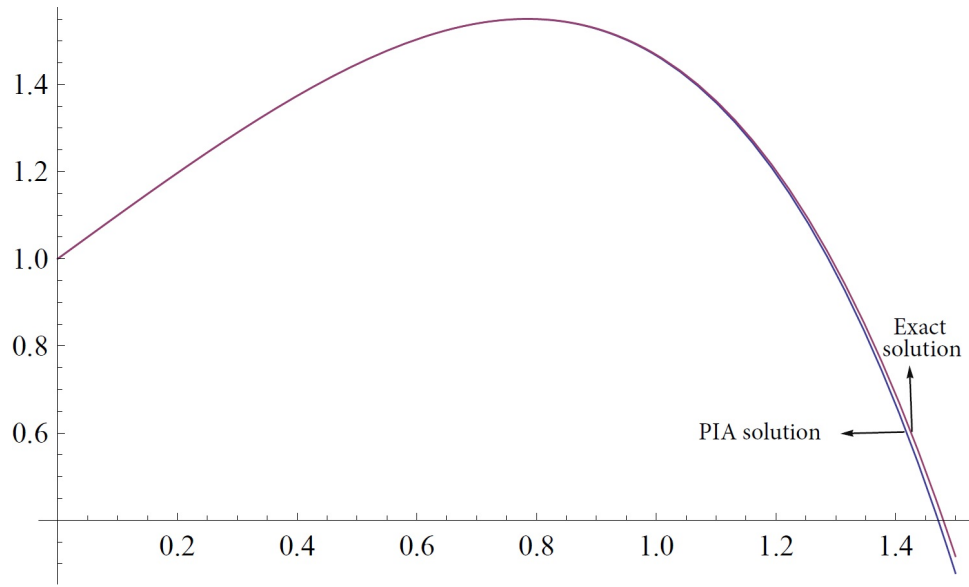


Figure 4: Comprasion of the exact solution and the PIA solution  $u_{2,5}(t)$  of Example 1. for  $\alpha = \beta = 1$ .

## 5 Authors contributions

All authors contributed in development of the manuscript and solving problems. All authors read and approved the final manuscript.

## 6 Competing interests

The authors declare that they have no competing interest.

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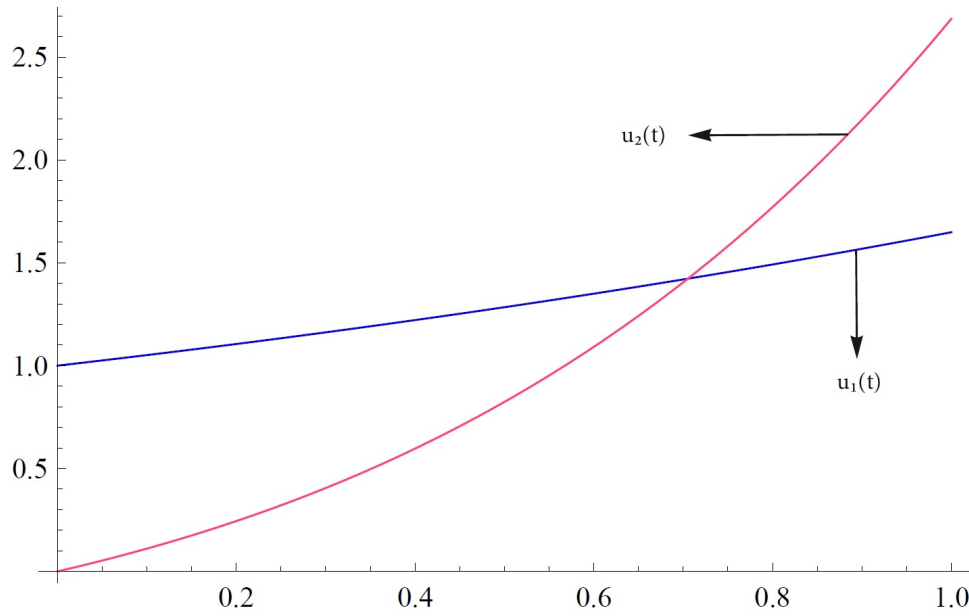


Figure 5: Comprasion of the  $u_{1,4}(t)$  and  $u_{2,4}(t)$  of Example 2. for  $\alpha_1 = \alpha_2 = 1$ .

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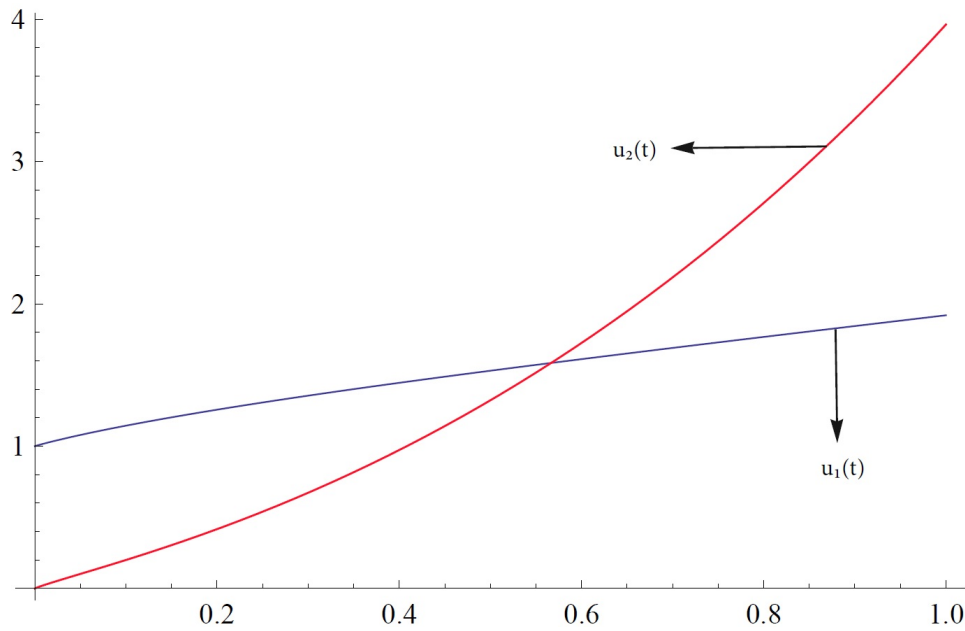


Figure 6: Comparison of the  $u_{1,4}(t)$  and  $u_{2,4}(t)$  of Example 2. for  $\alpha_1 = 0.5$  and  $\alpha_2 = 0.8$ .

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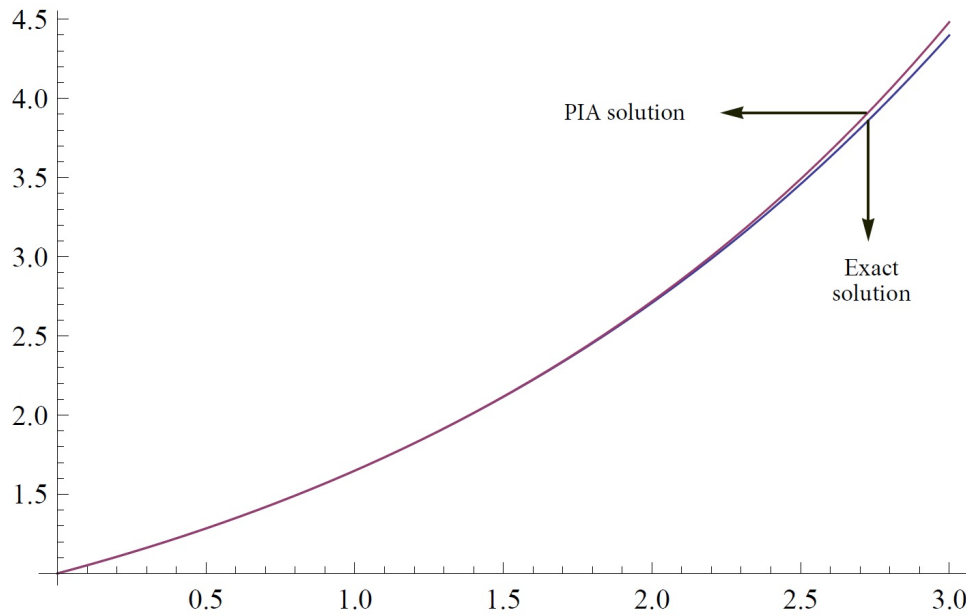


Figure 7: Comparison of the exact solution and the PIA solution  $u_{1,4}(t)$  of Example 2. for  $\alpha_1 = \alpha_2 = 1$ .

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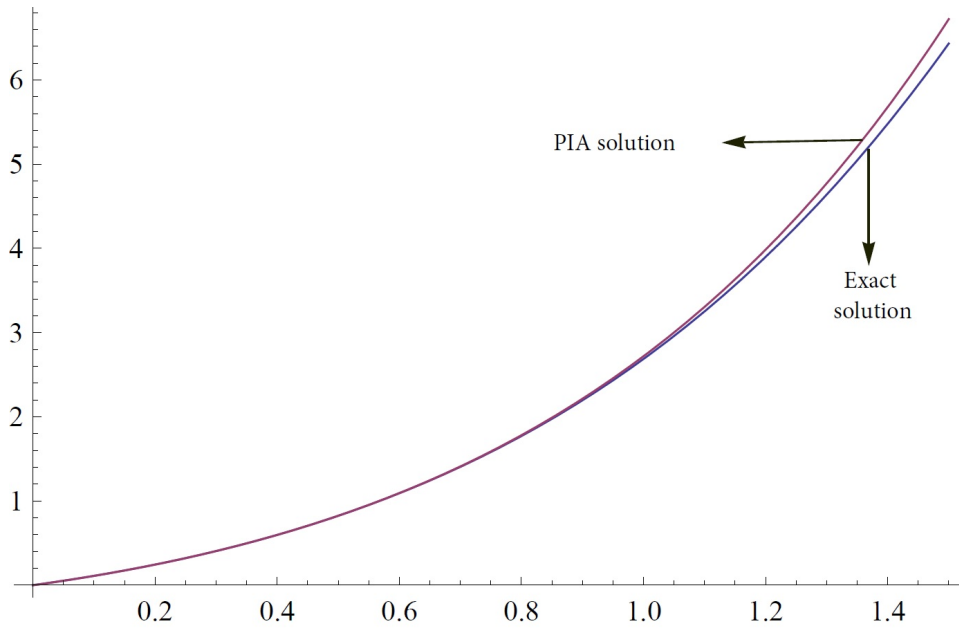


Figure 8: Comparison of the exact solution and the PIA solution  $u_{2,4}(t)$  of Example 2. for  $\alpha_1 = \alpha_2 = 1$ .